

Is having a unique equilibrium robust?

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Abstract

We investigate whether having a unique equilibrium (or a given number of equilibria) is robust to perturbation of the payoffs, both for Nash equilibrium and correlated equilibrium. We show that the set of n -player finite games with a unique correlated equilibrium is open, while this is not true of Nash equilibrium for $n > 2$. The crucial lemma is that a unique correlated equilibrium is a quasi-strict Nash equilibrium. Related results are studied. For instance, we show that generic two-person zero-sum games have a unique correlated equilibrium and that, while the set of symmetric bimatrix games with a unique symmetric Nash equilibrium is not open, the set of symmetric bimatrix games with a unique and quasi-strict symmetric Nash equilibrium is.

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[†]This paper was written during my PhD, supervised by Sylvain Sorin, at the laboratoire d'économétrie de l'Ecole polytechnique, Paris, and revised while at the Stockholm School of Economics. It was motivated by a related note of Noa Nitzan (2005). It greatly benefited from comments of Eilon Solan, Bernhard von Stengel, and participants to the game theory seminar of the Institut Henri Poincaré. Errors and shortcomings are mine.

The relevance of a phenomenon arising in a game often hinges upon this phenomenon being robust to perturbation of the payoffs of the game. To establish such robustness results typically requires proving that some of the properties of the game we initially considered are themselves robust; that is, that the set of games having these properties is open. We investigate here whether the set of finite normal-form games with a unique equilibrium is open, for Nash equilibrium, correlated equilibrium, and variants of Nash equilibrium such as symmetric Nash equilibrium in symmetric games.

For Nash equilibrium in two-player games, the question has been solved by Jansen (1981), who showed that the set of bimatrix games with a unique Nash equilibrium is open. However, this result does not extend to three-player games nor to symmetric equilibria of symmetric bimatrix games: counterexamples are given in section 3. Our main result is that, by contrast, for any number of players n , the set of n -player finite games with a unique correlated equilibrium is open. This generalizes an earlier result of Noa Nitzan (2005).

An intuitive explanation of the discrepancy between the results on Nash equilibrium and those on correlated equilibrium is as follows: the proof of the openness of the set of bimatrix games with a unique Nash equilibrium uses three ingredients : upper-semi-continuity of the equilibrium correspondence, an element of linearity in the structure of the set of equilibria (the set of Nash equilibria of a bimatrix game is a finite union of convex polytopes) and the fact that in bimatrix games, a unique Nash equilibrium is quasi-strict (Jansen, 1981); that is, for each player, no pure best response to the strategy of the other player is outside the support of her own strategy.

For Nash equilibrium, the last two ingredients are specific to two-player games, and the last one is also lacking for symmetric Nash equilibria of symmetric bimatrix games. This accounts for our negative results. By contrast, for any number of players, there is strong element of linearity in the structure of the set of correlated equilibria (this is a polytope). Furthermore, and this is the crucial lemma, it may be shown that for any number of players, a unique correlated equilibrium is a quasi-strict Nash equilibrium. This follows from the strong complementary property of linear programs and dual reduction arguments (Myerson, 1997).

The material is organized as follows: Definitions and notations are introduced in section 1. Openness of the set of games with a unique correlated equilibrium is proved in section 2. Section 3 groups remarks and related

results: We first show that generic zero-sum games have a unique correlated equilibrium and discuss the connections of our work with Nitzan's (2005). It is then shown that the set of 3-player games with a unique Nash equilibrium and the set of symmetric bimatrix games with a unique symmetric Nash equilibrium are not open, but that the set of symmetric bimatrix games with a unique and quasi-strict symmetric Nash equilibrium is. It is also shown that for any $n, k \geq 2$, the set of n -player games with k Nash equilibria (or k extreme points to the set of correlated equilibria) is not open, though for almost every game G , every game in a neighborhood of G has the same number of equilibria as G . We then study the structure of the set of Nash equilibria for bimatrix games at the relative boundary of the set of games with a unique correlated equilibrium. Finally, we show that the set of two-player zero-sum games in which one of the players has a unique optimal strategy is not open. This explains that the openness of the set of games with a unique correlated equilibrium cannot be deduced easily from Hart and Schmeidler's (1989) proof of existence of correlated equilibria.

1 Definitions and main result

Let G be a finite n -player game. $I = \{1, 2, \dots, n\}$ is the set of players, S^i the set of pure strategies of player i and $S^{-i} := \times_{j \in I \setminus \{i\}} S^j$. The utility function of player i is $U^i : S = \times_{i \in I} S^i \rightarrow \mathbb{R}$. As usual, U^i is extended multilinearly to the set of probability distributions over S . A pure strategy profile is denoted by $s = (s^i, s^{-i})$ and a mixed strategy profile by $\sigma = (\sigma^i, \sigma^{-i})$. The support of σ^i is denoted by

$$Supp(\sigma^i) := \{s^i \in S^i : \sigma^i(s^i) > 0\}$$

and the set of pure best-responses to σ^{-i} by

$$PBR(\sigma^{-i}) := \{s^i \in S^i, \forall t^i \in S^i, U^i(s^i, \sigma^{-i}) \geq U^i(t^i, \sigma^{-i})\}$$

Finally, for any finite set T , the simplex of probability distributions over T is denoted by $\Delta(T)$.

A *correlated strategy* of the players in I is a probability distribution over the set S of pure strategy profiles. Thus $\mu = (\mu(s))_{s \in S}$ is a correlated strategy if:

$$\mu(s) \geq 0 \quad \forall s \in S \tag{1}$$

$$\sum_{s \in S} \mu(s) = 1 \quad (2)$$

Henceforth, the conditions in (1) will be called *nonnegativity constraints*. For s^i, t^i in S^i and μ in $\Delta(S)$, let

$$h^{s^i, t^i}(\mu) := \sum_{s^{-i} \in S^{-i}} \mu(s) [U^i(s) - U^i(t^i, s^{-i})]$$

where, as throughout, $s = (s^i, s^{-i})$.

Definition. A correlated strategy μ is a correlated equilibrium (Aumann, 1974) if it satisfies the following incentive constraints:

$$h^{s^i, t^i}(\mu) \geq 0, \quad \forall i \in I, \forall s^i \in S^i, \forall t^i \in S^i \quad (3)$$

Since conditions (1), (2) and (3) are all linear in μ , it follows that the set of correlated equilibria of any finite game is a polytope.

A n -player finite game has size $m_1 \times m_2 \times \dots \times m_n$ if, for every i in I , the pure strategy set of player i has cardinal m_i . Assimilating a game and its payoff matrices, a n -player game of size $m_1 \times m_2 \times \dots \times m_n$ may be seen as a point in $(\mathbb{R}^{m_1 m_2 \dots m_n})^n$, hence the notions of a neighborhood of a game and of an open set of games. The main result of this paper is that:

Proposition 1. *The set of n -player games of size $m_1 \times m_2 \times \dots \times m_n$ with a unique correlated equilibrium is an open subset of the set of games of size $m_1 \times m_2 \times \dots \times m_n$. Furthermore, if a n -player finite game has a unique correlated equilibrium σ , then the (unique) correlated equilibrium of every nearby game has the same support as σ .*

2 Proof

Let G be a game with a unique correlated equilibrium and (G_n) a sequence of games converging towards G . We need to show that, for n large enough, the game G_n has a unique correlated equilibrium. The proof runs as follows: Let σ denote the unique correlated equilibrium of G . A dual reduction argument shows that σ is a quasi-strict Nash equilibrium (lemma 3). Together with the upper semi-continuity of the Nash equilibrium correspondence this implies that, for n large enough, G_n has a quasi-strict Nash equilibrium with

the same support as σ (lemma 5). Since two quasi-strict Nash equilibria with the same support satisfy the same nonnegativity and incentive constraints with equality (lemma 6), it follows that, for n large enough, G_n has a correlated equilibrium satisfying with equality the same constraints as σ . Due to a general result on polytopes (lemma 2), this implies that, for n sufficiently large, the correlated equilibrium polytope of G_n is a singleton. This completes the proof.

We begin with the result on polytopes: Let (\mathbf{A}_n) be a sequence of $p \times q$ real matrices, (\mathbf{b}_n) a sequence of column vectors of size p , and $J = \{1, \dots, p\}$. Let

$$C_n = \{\mathbf{x} \in \mathbb{R}^q, \mathbf{A}_n \mathbf{x} \geq \mathbf{b}_n\}$$

($\mathbf{A}_n \mathbf{x} \geq \mathbf{b}_n$ means that the weak inequality holds for each coordinate). Make the following assumptions: first, C_n is uniformly bounded:

$$\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, \forall \mathbf{x} \in C_n, \max_{j \in J} |(\mathbf{A}_n \mathbf{x})_j| \leq M \quad (4)$$

In particular, C_n is a polytope. Second, (\mathbf{A}_n) and (\mathbf{b}_n) converge respectively towards the matrix \mathbf{A} and the vector \mathbf{b} . Third, the “limit polytope”

$$C = \{\mathbf{x} \in \mathbb{R}^q, \mathbf{A} \mathbf{x} \geq \mathbf{b}\}$$

is a singleton: $C = \{\bar{\mathbf{x}}\}$. Let $J' := \{j \in J, (\mathbf{A} \bar{\mathbf{x}})_j = b_j\}$ denote the set of constraints binding at $\bar{\mathbf{x}}$, and let $J'' = J \setminus J'$. Finally, let $b_{n,j}$ denote the j^{th} component of \mathbf{b}_n .

Lemma 2. *If there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, there exists \mathbf{x}_n in C_n with*

$$\forall j \in J', (\mathbf{A}_n \mathbf{x}_n)_j = b_{n,j} \quad (5)$$

then for n large enough, C_n is a singleton.

Proof. We begin by showing that, for n large enough, all constraints satisfied with strict inequality by $\bar{\mathbf{x}}$ are satisfied with strict inequality by every \mathbf{x} in C_n :

$$\exists N' \in \mathbb{N}, \forall n \geq N', \forall \mathbf{x} \in C_n, \forall j \in J'', (\mathbf{A}_n \mathbf{x})_j > b_{n,j} \quad (6)$$

For all n in N , let $\mathbf{z}_n \in C_n$. To establish (6), it suffices to show that for n large enough,

$$\forall j \in J'', (\mathbf{A}_n \mathbf{z}_n)_j > b_{n,j} \quad (7)$$

Due to (4), the sequence (\mathbf{z}_n) is bounded. Furthermore, since $\mathbf{A}_n \rightarrow \mathbf{A}$ and $\mathbf{b}_n \rightarrow \mathbf{b}$, any accumulation point $\bar{\mathbf{z}}$ of (\mathbf{z}_n) satisfies $\bar{\mathbf{z}} \in C$, hence $\bar{\mathbf{z}} = \bar{\mathbf{x}}$. It follows that (\mathbf{z}_n) converges to $\bar{\mathbf{x}}$. Therefore $\mathbf{A}_n \mathbf{z}_n - \mathbf{b}_n$ converges to $\mathbf{A} \bar{\mathbf{x}} - \mathbf{b}$. Since $(\mathbf{A} \bar{\mathbf{x}})_j - \mathbf{b}_j > 0$ for every j in J'' , it follows that, for n large enough, (7) is satisfied. This completes the proof of (6).

We now prove the lemma: let $n \geq \max(N, N')$ and let \mathbf{x}_n satisfy (5). If C_n is not a singleton, then C_n has an extreme point $\mathbf{z}_n \neq \mathbf{x}_n$. By basic properties of extreme points, one of the constraints defining C_n is binding at \mathbf{z}_n but not at \mathbf{x}_n . Since $(\mathbf{A}_n \mathbf{x}_n)_j = b_{n,j}$ for all j in J' , this implies that there exists j in J'' such that $(\mathbf{A}_n \mathbf{z}_n)_j = b_{n,j}$. This contradicts (6). Therefore C_n is a singleton. \square

Lemma 3. *If an n -player finite game has a unique correlated equilibrium σ then this correlated equilibrium is a quasi-strict Nash equilibrium.*

Proof. The fact that σ is a Nash equilibrium follows from the existence of Nash equilibria and the fact that Nash equilibria are correlated equilibria. What we really need to show is that σ is quasi-strict.

For each player i in I , let α^i be a transition probability over the set of pure strategies of player i :

$$\begin{aligned} \alpha^i : S^i &\rightarrow \Delta(S^i) \\ s^i &\rightarrow \alpha^i * s^i \end{aligned}$$

A mixed strategy τ^i of player i is α^i -invariant if $\alpha^i * \tau^i = \tau^i$ where the mixed strategy $\alpha^i * \tau^i$ is defined by

$$[\alpha^i * \tau^i](t^i) = \sum_{s^i \in S^i} \tau^i(s^i) ([\alpha^i * s^i](t^i)) \quad \forall t^i \in S^i$$

It follows from (Nau and McCardle, 1990, last paragraph of section 2 and proposition 2) that there exists a vector of transition probabilities α such that, for every pure strategy profile s in S ,

$$\sum_{i \in I} [U^i(\alpha^i * s^i, s^{-i}) - U^i(s)] \geq 0 \quad (8)$$

with strict inequality if s has probability zero in all correlated equilibria. Fix such a vector $\alpha = (\alpha^i)_{i \in I}$. We claim that:

Claim 4. *For every i in I , the mixed strategy σ^i is α^i -invariant.*

This will be proved in the end. Assume that the pure strategy s^i does not belong to the support of σ^i and let $\tau = (s^i, \sigma^{-i})$. Since σ is the unique correlated equilibrium of G , it follows that every pure strategy profile s in the support of τ has probability zero in all correlated equilibria. Therefore, by definition of α ,

$$\tau(s) > 0 \Rightarrow \sum_{k \in I} [U^k(\alpha^k * s^k, s^{-k}) - U^k(s)] > 0$$

It follows that

$$\sum_{k \in I} [U^k(\alpha^k * \tau^k, \tau^{-k}) - U^k(\tau)] = \sum_{s \in S} \tau(s) \sum_{k \in I} [U^k(\alpha^k * s^k, s^{-k}) - U^k(s)] > 0 \quad (9)$$

(to prove the equality, use $U^k(\alpha^k * \tau^k, \tau^{-k}) = \sum_{s^k \in S^k} \tau^k(s^k) U^k(\alpha^k * s^k, \tau^{-k})$). For every $k \neq i$, $\tau^k = \sigma^k$ hence, by claim 4, τ^k is α^k -invariant. Therefore, (9) boils down to $U^i(\alpha^i * \tau^i, \tau^{-i}) > U^i(\tau)$; that is,

$$U^i(\alpha^i * s^i, \sigma^{-i}) > U^i(s^i, \sigma^{-i})$$

This implies that s^i is not a best-response to σ^{-i} . Since s^i was an arbitrary strategy not in the support of σ^i , it follows that σ is quasi-strict.

It only remains to prove claim 4. The proof is based on dual reduction (Myerson, 1997). Fix α as above. Note that, due to (8), α is a dual vector in the sense of Myerson (1997). Define the α -reduced game G/α as in (Myerson, 1997). That is, in G/α , the set of players and the payoffs are as in G , but the mixed strategies available to player i are only those mixed strategies σ^i of player i in G that are α^i -invariant. Myerson (1997) shows that G/α is a finite game, hence it has a Nash equilibrium. Let $\tilde{\sigma}$ be a Nash equilibrium of G/α . By definition of G/α , we may see $\tilde{\sigma}$ as a mixed strategy profile of G , with $\tilde{\sigma}^i$ α^i -invariant. Since $\tilde{\sigma}$ is a Nash equilibrium of G/α , it follows from theorem 1 of Myerson (1997) that $\tilde{\sigma}$ is a Nash equilibrium of G . Therefore $\tilde{\sigma} = \sigma$. Since $\tilde{\sigma}^i$ is α^i -invariant, this implies that σ^i is α^i -invariant too. This completes the proof. \square

The following lemma is a version of lemma 4.1 of Jansen (1981).

Lemma 5. *If the n -player game G has a unique Nash equilibrium σ and this Nash equilibrium is quasi-strict, then there exists a neighbourhood Ω_G of G such that, for every game \hat{G} in Ω_G and every Nash equilibrium $\hat{\sigma}$ of \hat{G} , the support of $\hat{\sigma}$ is equal to the support of σ and $\hat{\sigma}$ is quasi-strict.*

Proof. Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of games converging to G and σ_n a Nash equilibrium of G_n . To prove lemma 5, it is enough to show that, for n large enough, the support of σ_n is equal to the support of σ and σ_n is quasi-strict. Since the Nash equilibrium correspondence is upper semi-continuous and since G has a unique Nash equilibrium, it follows that σ_n converges to σ . Therefore,

$$\exists N \in \mathbb{N}, \forall n \geq N, \forall i \in I, \text{Supp}(\sigma^i) \subseteq \text{Supp}(\sigma_n^i) \quad (10)$$

Furthermore, if $U^i(s^i, \sigma^{-i}) < U^i(\sigma)$ then for n large enough, $U_n^i(s^i, \sigma_n^{-i}) < U_n^i(\sigma_n)$, where U_n^i denotes the utility function of player i in the game G_n . Therefore,

$$\exists N' \in \mathbb{N}, \forall n \geq N', \forall i \in I, PBR(\sigma_n^{-i}) \subseteq PBR(\sigma^{-i}) \quad (11)$$

Finally, since the Nash equilibrium σ is quasi-strict, it follows that $PBR(\sigma^{-i}) = \text{Supp}(\sigma^{-i})$. Together with (10) and (11), this implies that for n large enough:

$$\text{Supp}(\sigma_n^i) \subseteq PBR(\sigma_n^{-i}) \subseteq PBR(\sigma^{-i}) = \text{Supp}(\sigma^i) \subseteq \text{Supp}(\sigma_n^i)$$

Since the beginning and the end of this chain of inclusion are equal, this is a chain of equality. In particular, $\text{Supp}(\sigma_n^i) = PBR(\sigma_n^{-i}) = \text{Supp}(\sigma^i)$. The result follows. \square

Lemma 6. *Let G and \hat{G} be two games with the same set of players and strategies. Let σ and $\hat{\sigma}$ be Nash equilibria of, respectively, G and \hat{G} . Assume that σ and $\hat{\sigma}$ have the same support and are both quasi-strict. Then, among the nonnegativity and incentive constraints defining correlated equilibria, σ and $\hat{\sigma}$ satisfy the same constraints with equality.*

Proof. Since, by assumption, σ and $\hat{\sigma}$ have the same support, the nonnegativity constraints they satisfy with equality are the same. We now show that the incentive constraints they satisfy with equality are also the same. Since σ is a product distribution, it follows that

$$h^{s^i, t^i}(\sigma) = \sigma^i(s^i) [U^i(s^i, \sigma^{-i}) - U^i(t^i, \sigma^{-i})] \quad \forall i, \forall s^i, \forall t^i$$

Let $\tilde{S} = \times_i \tilde{S}^i$ denote the support of both σ and $\hat{\sigma}$. If $s^i \notin \tilde{S}^i$, then $\sigma^i(s^i) = 0$ hence $h^{s^i, t^i}(\sigma) = 0$ for every t^i in \tilde{S}^i . If $s^i \in \tilde{S}^i$ and $t^i \in \tilde{S}^i$, then, since σ is a Nash equilibrium, $U^i(s^i, \sigma^{-i}) = U^i(t^i, \sigma^{-i})$ hence $h^{s^i, t^i}(\sigma) = 0$. Finally, if

$s^i \in \tilde{S}^i$ and $t^i \notin \tilde{S}^i$ then $\sigma^i(s^i) > 0$ and, since σ is quasi-strict, $U^i(s^i, \sigma^{-i}) - U^i(t^i, \sigma^{-i}) > 0$. Therefore, $h^{s^i, t^i}(\sigma) > 0$. Grouping these observations we obtain that $h^{s^i, t^i}(\sigma) > 0$ if and only if $s^i \in \tilde{S}^i$ and $t^i \notin \tilde{S}^i$. The same result holds for $\hat{\sigma}$ so that, letting $(\hat{h}^{s^i, t^i})_{s^i \in S^i, t^i \in S^i}$ denote the linear forms associated with the correlated equilibrium incentive constraints of \hat{G} , we have:

$$h^{s^i, t^i}(\sigma) = 0 \Leftrightarrow \hat{h}^{s^i, t^i}(\hat{\sigma}) = 0$$

This completes the proof. \square

We now conclude. Let G be a game with a unique correlated equilibrium σ and (G_n) be a sequence of games converging towards G . Let C_n be the correlated equilibrium polytope of G_n . Combining lemmas 3, 5 and 6, we obtain that, for n large enough, G_n has a correlated equilibrium σ_n satisfying with equality the same constraints as σ . By lemma 2, this implies that for n large enough, C_n is a singleton. This completes the proof of proposition 1.

3 Remarks and related results

1. The fact that the set of $m_1 \times m_2 \times \dots \times m_n$ games with a unique correlated equilibrium is nonempty is obvious: any dominance solvable game has a unique correlated equilibrium. Note also that generic two-player zero-sum games have a unique correlated equilibrium. This follows from the observation that: (i) a zero-sum game has a unique correlated equilibrium if and only if it has a unique Nash equilibrium (Forges, 1990); (ii) generic zero-sum games have a unique Nash equilibrium (Bohnenblust et al, 1950).

2. Nitzan (2005) proved independently and earlier a weaker version of proposition 1. More precisely, she proved that if a two-player $m \times m$ game has a unique correlated equilibrium and that this correlated equilibrium has full support, then every nearby game has a unique correlated equilibrium and this correlated equilibrium has full support. To prove this result with our method, it suffices to note that if a game has a unique and completely mixed Nash equilibrium, then every nearby game has a completely mixed Nash equilibrium, and then to apply lemma 2. This illustrates a difference between our arguments and Nitzan's: while she uses a theorem of the alternative, we do not need any theorem of the alternative to prove her results. (We do however use a theorem of the alternative to prove proposition 1. Indeed, the proof of lemma 3 uses Nau and McCardle's (1990) characterization

of strategy profiles with positive probability in at least one correlated equilibrium, which itself relies on a theorem of the alternative.)

3. The set of 3-player games with a unique Nash equilibrium is not open. The following $2 \times 2 \times 2$ counter-example is adapted from (Flesch et al, 1997) and was provided by Eilon Solan (personal communication).

$$\begin{pmatrix} 1, 1, 1 & 0, 1, 1 \\ 1, 1, 0 & 1, 0, 1 \end{pmatrix} \quad \begin{pmatrix} 1, 0, 1 - \epsilon & 1, 1, 0 \\ 0, 1, 1 & 0, 0, 0 \end{pmatrix} \quad (12)$$

Player 1 chooses a row (Top or Bottom), player 2 a column (Left or Right) and player 3 a matrix (West or East). For $\epsilon = 0$, there is a unique Nash equilibrium, in which all players play their first strategy (this will be proved below). However, for $\epsilon > 0$, there is a continuum of Nash equilibria. Indeed, every (partially) mixed strategy profile in which player 1 plays Bottom with probability less than $\epsilon/(1 + \epsilon)$ and player 2 and 3 stick to their first strategy is a Nash equilibrium. Thus, in 3-player games, there are sequences of games with a continuum of Nash equilibria converging towards a game with a unique Nash equilibrium.

The game (12) with $\epsilon = 0$ also provides an example of a game with a unique Nash equilibrium that is not quasi-strict. This calls for two remarks: first, while it is well known that 3-player games need not have a quasi-strict equilibrium, the counter-examples I found in the literature, e.g. (Raghavan, 2002), are of games with several Nash equilibria. Thus, up to my knowledge, whether a unique Nash equilibrium is necessarily quasi-strict was still open. Second, in two-player games, a unique Nash equilibrium is necessarily quasi-strict, as shown by Jansen (1981), and as also follows from the fact that every bimatrix game has a quasi-strict Nash equilibrium (Norde, 1999).

Proof that the game (12) with $\epsilon = 0$ has a unique Nash equilibrium: for $\epsilon = 0$, the game (12) may be described as follows: player $i + 1$ (counted modulo 3) wants to mismatch player i , except if all players play their first strategy. Thus, in a hypothetical equilibrium different from Top-Left-West, if i plays in pure strategy, then $i + 1$ must mismatch i , $i + 2$ mismatch $i + 1$ and $i + 3 = i$ mismatch $i + 2$; therefore, i must mismatch itself, a contradiction. It follows that the only equilibrium in which one of the players plays in pure strategy is Top-Left-West.

It remains to show that there are no completely mixed Nash equilibria. By contradiction, let $x \in]0, 1[$ (resp. y, z) be the probability of Bottom (resp. Right, East) in a hypothetical completely mixed Nash equilibrium. Since player 1 is indifferent between Top and Bottom, we have $y(1 - z) = (1 - y)z + yz = z$, hence $y > z$. Since the game is cyclically symmetric, it follows that $y > z > x > y$, which cannot be. This completes the proof. ■

4. A corollary of lemma 5 is that:

Corollary 7. *The set of games with a unique and strict Nash equilibrium is open.*

Indeed, if a game has a unique and strict Nash equilibrium σ , then by lemma 5, every Nash equilibrium of every nearby game has the same support as σ , hence is equal to σ as σ is pure. I do not know whether the set of games with a unique and *quasi*-strict Nash equilibrium is open.

5. The following example shows that, within the set of two-person symmetric games, the set of games with a unique symmetric Nash equilibrium is not open:

$$\begin{pmatrix} -\epsilon, -\epsilon & 1, 0 & 1, 0 \\ 0, 1 & 0, 0 & -1, -1 \\ 0, 1 & -1, -1 & 0, 0 \end{pmatrix}$$

For $\epsilon = 0$, this game has a unique symmetric Nash equilibrium: Top-Left. For $\epsilon > 0$, it has 3 symmetric Nash equilibria: $(\frac{1}{1+\epsilon}, \frac{\epsilon}{1+\epsilon}, 0)$, $(\frac{1}{1+\epsilon}, 0, \frac{\epsilon}{1+\epsilon})$, $(\frac{3}{3+2\epsilon}, \frac{\epsilon}{3+2\epsilon}, \frac{\epsilon}{3+2\epsilon})$. This is linked to the fact that, for $\epsilon = 0$, the unique symmetric Nash equilibrium is not quasi-strict. Indeed, the openness of the set of bimatrix games with a unique Nash equilibrium has the following analogue for symmetric games:

Proposition 8. *Within the set of two-person symmetric games, the set of two-person symmetric games with a unique symmetric Nash equilibrium and such that this Nash equilibrium is quasi-strict is open.*

Proof. Let G be a two-person game with a unique symmetric Nash equilibrium, with support $\tilde{S} = \tilde{S}^1 \times \tilde{S}^2$ (To make things clear: there might be other, asymmetric Nash equilibria). Assume that the unique symmetric equilibrium

is quasi-strict. It follows from a variant of lemma 5 that, within the set of two-person symmetric games, there exists a neighborhood Ω_G of G such that, for any game G' in Ω_G , any symmetric Nash equilibrium of G' has support \tilde{S} and is quasi-strict.

Fix G' in Ω_G . Since G' is symmetric, it has a symmetric Nash equilibrium σ . To establish proposition 8, it is enough to show that G' has no other symmetric Nash equilibrium. By contradiction, assume that G' has a symmetric Nash equilibrium $\tau \neq \sigma$. For every λ in \mathbb{R} , define the symmetric mixed strategy profile σ_λ by $\sigma_\lambda^i = \lambda\tau^i + (1 - \lambda)\sigma^i$, for $i = 1, 2$.

There are five types of incentive and nonnegativity constraints that σ_λ must satisfy in order to be a (symmetric) Nash equilibrium:

- (i) $\sigma_\lambda(s) \geq 0, s \in \tilde{S}$;
- (ii) $\sigma_\lambda(s) \geq 0, s \notin \tilde{S}$;
- (iii) $h^{s^i, t^i}(\sigma_\lambda) \geq 0, s^i \in \tilde{S}^i, t^i \notin \tilde{S}^i, i = 1, 2$;
- (iv) $h^{s^i, t^i}(\sigma_\lambda) \geq 0, s^i \in \tilde{S}^i, t^i \in \tilde{S}^i, i = 1, 2$;
- (v) $h^{s^i, t^i}(\sigma_\lambda) \geq 0, s^i \notin \tilde{S}^i, t^i \in \tilde{S}^i, i = 1, 2$.

Using the fact that both σ and τ are Nash equilibria with support \tilde{S} , it is easily checked that for every λ in \mathbb{R} , σ_λ satisfies (with equality) all constraints of types (ii), (iv) and (v).

Moreover, since $\sigma \neq \tau$ and since the set of Nash equilibria is compact, it follows that there exists a maximal value of λ such that σ_λ is a (symmetric) Nash equilibrium. Call this value λ_{max} . Since all symmetric Nash equilibria of G' have support \tilde{S} and are quasi-strict, they all satisfy with strict inequality all constraints of types (i) and (iii), hence so does $\sigma_{\lambda_{max}}$. Therefore, there exists $\lambda > \lambda_{max}$ such that σ_λ satisfies all constraints of type (i) and (iii). Since, as mentioned in the previous paragraph, σ_λ also satisfies all other constraints, it follows that σ_λ is a (symmetric) Nash equilibrium, contradicting the maximality of λ_{max} . \square

Finally, a variant of the proof of corollary 7 shows that the set of n -player symmetric games with a unique symmetric Nash equilibrium and such that this equilibrium is strict is open.

6. Until now, we focused on games with a unique equilibrium, but we might also ask whether, for $k \geq 2$, having k Nash equilibria, or k extreme points to the set of correlated equilibria, is a robust property. The answer is negative:

Proposition 9. *For every $n, k \geq 2$, the set of n -player games with k ($\leq k$, $\geq k$) Nash equilibria is not open; similarly, the set of n -player games with k ($\leq k$, $\geq k$) extreme points of the set of correlated equilibria is not open.*

We provide a counterexample for the case $n = 2$, $k = 3$. The counterexample is easily generalized to any numbers $n, k \geq 2$. For simplicity, call extreme correlated equilibria the extreme points of the set of correlated equilibria. Consider the following game:

$$\begin{array}{c} \begin{array}{ccc} & L & M & R \\ \begin{array}{c} T \\ M \\ B \end{array} & \left(\begin{array}{ccc} 0, 0 & -1, -1 & -1, 0 \\ -1, -1 & \epsilon, \epsilon & -1, 0 \\ 0, -1 & 0, -1 & 0, 0 \end{array} \right) \end{array} \quad (13)$$

For $\epsilon = 0$, both players can guarantee 0 by playing the third strategy, and this is the highest payoff they can get. It follows that in any correlated equilibrium, the off-diagonal strategy profiles have probability zero. Therefore, the three pure strategy profiles on the diagonal are the only Nash equilibria of the game and these are also the extreme correlated equilibria. For $\epsilon < 0$, the Nash equilibrium (M, M) disappears, and only two Nash equilibria remain, which again are also the extreme correlated equilibria. For $\epsilon > 0$, there are 4 Nash equilibria (resp. 6 extreme correlated equilibria): (T, L) and the Nash equilibria (resp. extreme correlated equilibria) of the 2×2 coordination game obtained by eliminating the strategies T and L .

7. The reason why a slight perturbation of the payoffs of (13) may alter the number of equilibria of the game is that not all equilibria are quasi-strict. Indeed, Jansen (1981, lemma 8.3 and remark 8.8) showed that if all equilibria of a bimatrix game are quasi-strict, then (i) the game has a finite number of equilibria and (ii) every nearby game has the same number of equilibria.

Using the fact that a finite game has a finite number of equilibrium components and upper-semi-continuity of the Nash equilibrium correspondence, it is easy to generalize Jansen's result as follows: if all equilibria of a n -player game are strongly stable (in the sense of van Damme, 1991) then (i) and (ii) above hold. Since for bimatrix games, all equilibria are strongly stable if and only if all equilibria are quasi-strict, this indeed generalizes Jansen's result. Furthermore, since for almost all games, all equilibria are strongly stable

(see, e.g., van Damme, 1991), it follows that (i) and (ii) hold for almost all games.

Finally, it follows from the above discussion, and it is easy to show directly that the set of n -player games with k equilibria, all strict, is open; but for $k \geq 2$, this set is actually void. Indeed, if a game has $k \geq 2$ equilibria, all strict, then it follows from an index argument that there exists at least $k - 1$ mixed Nash equilibria, a contradiction. See corollary 2 and theorem 3 of Ritzberger (1994).

8. In bimatrix games, both the set of games with a unique Nash equilibrium and the set of games with a unique correlated equilibrium are open. Since there are games with a unique Nash equilibrium but many correlated equilibria, the latter set is included in the former. The following examples show that on the relative boundary of the set of bimatrix games with a unique correlated equilibrium, there are games with a continuum of Nash equilibria, games with a finite number (> 1) of Nash equilibria, and games with a unique Nash equilibrium:

$$\begin{pmatrix} 0 \\ \epsilon \end{pmatrix} \quad \begin{pmatrix} 1, 1 & 0, 0 \\ 0, 0 & -\epsilon, -\epsilon \end{pmatrix} \quad \left(\begin{array}{ccc|c} 0, 0 & 2, 1 & 1, 2 & -1, x \\ 1, 2 & 0, 0 & 2, 1 & -1, x \\ 2, 1 & 1, 2 & 0, 0 & -1, x \\ \hline x, -1 & x, -1 & x, -1 & 0, 0 \end{array} \right)$$

The left game is a one-person game. For $\epsilon > 0$ it has a unique correlated equilibrium. For $\epsilon = 0$ it has a continuum of Nash equilibria. The middle game has a unique correlated equilibrium (Top-Left) for $\epsilon > 0$, but two Nash equilibria for $\epsilon = 0$. The game on the right is adapted from (Nau and McCardle, 1990, example 4). The 3×3 game in the top-left corner is due to Moulin and Vial (1978). This 3×3 game has a unique Nash equilibrium: $(1/3, 1/3, 1/3)$ for both players, with payoff 1; but putting probability $1/6$ on every off-diagonal square yields a correlated equilibrium with payoff $3/2$. Now consider the whole 4×4 game. For any value of x , $(4, 4)$ is a Nash equilibrium. For $1 < x \leq 3/2$, this is the unique Nash equilibrium, but not the unique correlated equilibrium (the correlated equilibrium with payoff $3/2$ of the 3×3 top-left game induces a correlated equilibrium of the whole game). For $x > 3/2$, this is the unique correlated equilibrium.

9. In order to prove the existence of correlated equilibria without using a fixed point theorem, Hart and Schmeidler (1989) associate to every finite game G an auxiliary zero-sum game whose size depends only on the size of G and whose payoff matrix depends continuously on the payoff matrices of G . In this auxiliary zero-sum game, the optimal strategies of the maximizer correspond exactly to the correlated equilibria of G , so that G has a unique correlated equilibrium if and only if, in the auxiliary game, the maximizer has a unique optimal strategy. Therefore, to prove the openness of the set of games with a unique correlated equilibrium, it would have been enough to show that: *If in a two-player zero-sum game, one of the players has a unique optimal strategy, then in every nearby zero-sum game this player has a unique optimal strategy.*

However, this turns out to be false: let G_ϵ be the two-player zero-sum game with payoff matrix for the row player

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \quad \begin{array}{cc} L & R \\ \left(\begin{array}{cc} -\epsilon & 0 \\ 0 & -1 \\ 0 & -1 \end{array} \right) \end{array}$$

For $\epsilon = 0$, the row player has a unique optimal strategy (playing T). But for $\epsilon > 0$, the row player has an infinite number of optimal strategies: playing T with probability $1/(1 + \epsilon)$ and playing M and B with any probabilities summing to $\epsilon/(1 + \epsilon)$.

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